J Appl Math Comput (2008) 27: 293–305
DOI 10.1007/s12190-008-0044-5

JAMC

Stability and convergence of difference scheme for nonlinear evolutionary type equations

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Received: 10 March 2007 / Revised: 22 June 2007 / Published online: 18 March 2008

Abstract A finite difference scheme is derived for the initial-boundary problem for the nonlinear equation system

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u),$$

where A is a complex diagonal matrix, f is a complex vector function. The stability and convergence in discrete L^∞-norm of proposed Crank-Nicolson type finite difference schemes is proved. No restrictions on the ratio of time and space grid steps are assumed. Some numerical experiments have been conducted in order to validate the theoretical results.

Keywords Nonlinear evolutionary type equations · Crank-Nicolson type difference scheme · Existence · Stability · Uniqueness · Convergence

Mathematics Subject Classification (2000) 65M06 · 65M15 · 65N30

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1 Introduction

Two classes of nonlinear non stationary equations are considered: the Kuramoto-Tsuzuki equation describes the behavior of many two-component systems in a neighborhood of the bifurcation point [13]. Reaction-diffusion type equations have been applied in the study of broad class of nonlinear processes, including a well-known synergetic model [2, 14].

The problem of constructing and validating difference schemes for these classes of problems has been in detail taken up in [9, 10], see also [4, 7, 11, 21, 22] and [23]. A finite element Galerkin method had been discussed in [16, 17].

We consider the initial-boundary value problem for the nonlinear evolution problem:

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \Omega,\ 0 < t \leq T, \tag{1.1a}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ 0 < t \leq T, \tag{1.1b}$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \tag{1.1c}$$

where $u = (u_1, u_2, \ldots, u_r)$ is a complex vector-function, A is a complex diagonal matrix, f is a complex vector-function, u_0 is a given complex vector function, $\Omega = (0, 1)$ and $T > 0$.

Throughout this paper, we will be interested in the following cases:

if $\mathrm{Im}(A) = 0$ we have the reaction diffusion type equation,

if $\mathrm{Re}(A) > 0$ and $\mathrm{Im}(A) \neq 0$ we have the Kuramoto-Tsuzuki equation.

Observe that the study of system (1.1), in view of the diagonality of the matrix A, is really no different than the study of a single equation. Henceforth, we shall consider the single equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \Omega,\ 0 < t \leq T, \tag{1.2a}$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \leq T, \tag{1.2b}$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}. \tag{1.2c}$$

The following assumptions are made.

$$\text{(i)} \quad a \in \mathbb{C}, \quad \text{and} \quad \mathrm{Re}(a) = \alpha > 0, \tag{1.3}$$

$$\text{(ii)} \quad f(u) = b|u|^2 u, \tag{1.4}$$

$$\text{(iii)} \quad b \in \mathbb{C}, \quad \text{and} \quad \mathrm{Re}(b) = \beta < 0. \tag{1.5}$$

The condition (1.3) means the positivity of the heat conduction coefficient. Tsertsadze studied in [22] the convergence of difference schemes for the Kuramoto-Tsuzuki equation and for systems of reaction-diffusion type. For the mixed boundary value problem of Kuramoto-Tsuzuki equation, he constructed a difference scheme, based on Crank-Nicolson method, and proved that the latter is convergent in energy norm (L^2-norm) with the convergence rate of order $O(k^2 + h^{\frac{3}{2}})$ if $k = O(h^{2+\varepsilon})$ ($\varepsilon > 0$), the problem being nonlinear. In other words, the scheme is conditionally stable. Considering the same problem, but inhomogeneous equation, Sun suggested in [20] another scheme that is convergent in energy norm with convergence rate of order $O(k^2 + h^2)$ if $k = O(h^{\frac{1}{4}+\varepsilon})$ ($\varepsilon > 0$). Sun's linearized scheme is also conditionally stable. In [5], Cao and Sun studied the numerical solution of initial-Neumann problem for a strongly coupled reaction-diffusion system where the matrix A realizing the coupling is not symmetric. Their proposed scheme is convergent under a Lipschitz condition on the function $f(u)$ defining the nonlinear term. It leads to a second order convergence, but in L^2-norm.

In this paper, a non linearized finite difference scheme, based in Crank-Nicolson method provides a second order convergence in L^∞-norm. Besides, the difference scheme is unconditionally stable.

In Sect. 2, the Crank-Nicolson scheme for (1.2) is derived and preliminary notations are introduced. In Sect. 3, the existence of the difference solution is proved by virtue of the Brouwer fixed point theorem. In Sect. 4, some a priori estimates for numerical solutions are achieved; the uniqueness of the difference solution is also proved. The Sect. 5 is devoted to the study of the convergence of our difference scheme. The numerical experiments conducted in Sect. 6 confirm with the theoretical analysis.

2 Discretizations

Let N, J be any positif integers and $h = \frac{1}{J+1}$, $k = \frac{T}{N}$, $x_j = jh$, $j = 0, 1, \ldots, J + 1$ and $t_n = nk$, $t_{n+\frac{1}{2}} = (n + \frac{1}{2})k$, $u_j^n = u(x_j, t_n)$, $u^n = (u_0^n, u_1^n, \ldots, u_{J+1}^n)$ for $n = 0, 1, \ldots, N$.

Let $S = \{v = (v_0, v_1, \ldots, v_{J+1}) \in \mathbb{C}^{J+2},\ v_0 = v_{J+1} = 0\}$. We define the difference operators as follows, for a function $v \in S$,

$$\nabla_h^- v_j = \frac{v_j - v_{j-1}}{h}, \qquad \nabla_h^+ v_j = \frac{v_{j+1} - v_j}{h},$$

$$\Delta_h v_0 = \Delta_h v_{J+1} = 0, \qquad \Delta_h v_j = \frac{1}{h^2}(v_{j+1} - 2v_j + v_{j-1}), \quad 1 \le j \le J,$$

$$v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}, \qquad \partial_t v^n = \frac{v^{n+1} - v^n}{k}.$$

For $v, w \in S$, we define

$$(v, w)_h = h \sum_{j=1}^{J} v_j \overline{w}_j, \qquad |v|_{1,h} = \left[h \sum_{j=0}^{J} |\nabla_h^+ v_j|^2 \right]^{\frac{1}{2}},$$

$$\|v\|_h = \|v\|_{2,h} = \left[h \sum_{j=1}^{J} |v_j|^2 \right]^{\frac{1}{2}}, \qquad \|v\|_{p,h} = \left[h \sum_{j=1}^{J} |v_j|^p \right]^{\frac{1}{p}}, \quad p \geq 1.$$

Then, we may easily see that the following relations hold as in [3], for $v, w \in S$

$$(\Delta_h v, w)_h = -h \sum_{j=0}^{J} (\nabla_h^+ v_j)(\nabla_h^+ \overline{w}_j), \tag{2.1}$$

$$-(\Delta_h v, v)_h = |v|_{1,h}^2. \tag{2.2}$$

Using the above notations, we discretize the problem (1.2a)–(1.2c) by the following Crank-Nicolson-type finite difference scheme and define approximations $U^n \in S$ of u^n recursively by:

$$\partial_t U_j^n = a \Delta_h U_j^{n+\frac{1}{2}} + b\varphi\big(U_j^{n+\frac{1}{2}}\big), \quad 1 \leq j \leq J,\ 0 \leq n \leq N-1, \tag{2.3a}$$

$$U_j^0 = u_0(x_j), \quad 1 \leq j \leq J, \tag{2.3b}$$

where $\varphi(z) = |z|^2 z$.

3 Existence

In this section, we shall use the following Brower fixed-point theorem [3] in order to show the existence of solutions for the Crank-Nicolson finite difference scheme (2.3).

Theorem 1 *Let H be a finite dimensional space with inner product $(\cdot,\cdot)_H$, and norm $\|\cdot\|_H$. Let the map $g : H \to H$ be continuous. Suppose that there exists $\lambda > 0$ such that* $\mathrm{Re}(g(Z), Z)_H \geq 0$ *for all Z with $\|Z\|_H = \lambda$. Then there exists $Z^* \in H$ such that $g(Z^*) = 0$ and $\|Z^*\| \leq \lambda$.*

Theorem 2 *The solution U^n of the Crank-Nicolson finite difference scheme* (2.3a)–(2.3b) *exists.*

Proof In order to prove Theorem 2 by mathematical induction, we assume that $U^0, U^1, \ldots, U^n$ exist for $n < N$. Let g be the function defined by

$$g(V) = V - U_j^n - \frac{ak}{2}\Delta_h V - \frac{bk}{2}\varphi(V). \tag{3.1}$$

Then g is obviously continuous.

Taking in (3.1) the inner product with V, we obtain from (2.2)

$$(g(V), V)_h = \|V\|_h^2 - (U^n, V)_h + \frac{k}{2}\big[a|V|_{1,h}^2 - b\|V\|_{4,h}^4\big].$$

Therefore,

$$\operatorname{Re}(g(V), V)_h = \|V\|_h^2 - \operatorname{Re}(U^n, V)_h + \frac{k}{2}\big[\alpha|V|_{1,h}^2 - \beta\|V\|_{4,h}^4\big].$$

Using the assumptions (1.3) and (1.5), we find

$$\begin{aligned}\operatorname{Re}(g(V), V)_h &\geq \|V\|_h^2 - \|U^n\|_h\|V\|_h \\ &= \|V\|_h[\|V\|_h - \|U^n\|_h].\end{aligned}$$

Hence, it is obvious that $\operatorname{Re}(g(V), V)_h > 0$ for $\|V\|_h = \|U^n\|_h + 1$. It follows from Theorem 1 that there exists $V^* \in S$ such that $g(V^*) = 0$. If we take $U^{n+1} = 2V^* - U^n$, then U^{n+1} satisfies (2.3). This completes the proof of Theorem 2. □

4 Stability and uniqueness

Below, we shall give some a priori estimates of difference solutions.

4.1 Auxiliary lemmas

Lemma 1 *Suppose hypothesis* (1.3), (1.4) *and* (1.5) *are satisfied. Then, for the solution of the problem* (2.3) *the following estimate holds*:

$$\|U^n\|_h \leq \|U^0\|_h, \quad 1 \leq n \leq N.$$

Proof Taking the inner product of (2.3a) with $U^{n+\frac{1}{2}}$ and using (2.2), we find

$$(\partial_t U^n, U^{n+\frac{1}{2}}) = -a\big|U^{n+\frac{1}{2}}\big|_{1,h}^2 + b\big\|U^{n+\frac{1}{2}}\big\|_{4,h}^4.$$

Taking the real part, we find from (1.3) and (1.5)

$$\frac{1}{2k}\big(\|U^{n+1}\|_h^2 - \|U^n\|_h^2\big) \leq 0.$$

Therefore,

$$\|U^{n+1}\|_h^2 \leq \|U^n\|_h^2.$$

This completes the proof. □

Next, we will use the following Lemma (see [25]).

Lemma 2 *For $v \in S$, we have for $p \geq 2$*

$$\|v\|_{p,h} \leq C\big[\|v\|_h^{1-a}|v|_{1,h}^a + \|v\|_h\big],$$

where $a = \frac{p-2}{2p}$ and C is a positive constant independent of p and h.

4.2 Stability

We now show that the solution U^n of (2.3) is bounded in L^∞-norm.

Theorem 3 *Let U^n be solution of (2.3). Suppose that $u_0 \in H_0^1(0,1)$. Then, there exists a constant $\widetilde{C}$ independent of h and k such that for k sufficiently small*

$$\|U^n\|_{\infty,h} \leq \widetilde{C}.$$

Proof Multiplying (2.3a) by $\frac{1}{a}$ and forming its inner product with $\partial_t U^n$, from (2.1) it follows that

$$\frac{1}{a}\|\partial_t U^n\|_h^2 = -h\sum_{j=0}^{J}\nabla_h^+\big(U_j^{n+\frac{1}{2}}\big)\nabla_h^+(\overline{\partial_t U^n}) + \frac{b}{a}\big(\varphi(U^{n+\frac{1}{2}}), \partial_t U^n\big)_h.$$

Taking the real part, we obtain

$$\begin{aligned}\frac{\alpha}{|a|^2}\|\partial_t U^n\|_h^2 &= -\frac{1}{2k}\big(|U^{n+1}|_{1,h}^2 - |U^n|_{1,h}^2\big) + \mathrm{Re}\Big[\frac{b}{a}\big(\varphi(U^{n+\frac{1}{2}}), \partial_t U^n\big)_h\Big]\\ &\leq -\frac{1}{2k}\big(|U^{n+1}|_{1,h}^2 - |U^n|_{1,h}^2\big) + \frac{|b|}{|a|}\|U^{n+\frac{1}{2}}\|_{6,h}^3\|\partial_t U^n\|_h\\ &\leq -\frac{1}{2k}\big(|U^{n+1}|_{1,h}^2 - |U^n|_{1,h}^2\big) + \frac{|b|^2}{4\alpha}\|U^{n+\frac{1}{2}}\|_{6,h}^6 + \frac{\alpha}{|a|^2}\|\partial_t U^n\|_h^2.\end{aligned}$$

Therefore, we have

$$\frac{1}{k}\big(|U^{n+1}|_{1,h}^2 - |U^n|_{1,h}^2\big) \leq \frac{|b|^2}{2\alpha}\|U^{n+\frac{1}{2}}\|_{6,h}^6. \tag{4.1}$$

Noting that

$$(x_1+x_2)^6 \leq 2^5(x_1^6+x_2^6), \quad x_1 \geq 0,\ x_2 \geq 0. \tag{4.2}$$

Applying Lemma 2 with $p=6$, and using (4.2), we obtain

$$\|U^{n+\frac{1}{2}}\|_{6,h}^6 \leq C\big[\|U^{n+\frac{1}{2}}\|_h^4|U^{n+\frac{1}{2}}|_{1,h}^2 + \|U^{n+\frac{1}{2}}\|_h^6\big].$$

Substituting the above inequality in (4.1) and using Lemma 1, we find

$$\frac{1}{k}\big(|U^{n+1}|_{1,h}^2 - |U^n|_{1,h}^2\big) \leq C\big(|U^{n+1}|_{1,h}^2 + |U^n|_{1,h}^2\big) + C',$$

where C and C' are two constants depending of $\|U^0\|_h$, $|b|$ and α.

For small k, by Gronwall's discrete inequality [6, 8, 24], we find

$$|U^n|_{1,h}^2 \leq C(T)|U^0|_{1,h}^2 + C'(T). \tag{4.3}$$

It follows from Lemma 2 with $p=\infty$, and Lemma 1 and (4.3) that

$$\|U^n\|_{\infty,h} \le \widetilde{C},$$

where $\widetilde{C} = \widetilde{C}(u_0, T, |b|, \alpha)$; this completes the proof. □

4.3 Uniqueness

Theorem 4 *Assume that the initial condition $u_0 \in H_0^1(0,1)$. Then, for k small enough the solution of* (2.3a)–(2.3b) *exists uniquely.*

Proof Let V^n be another solution of (2.3). Then V^n satisfies

$$\partial_t V^n = a\Delta_h V^{n+\frac{1}{2}} + b\varphi(V^{n+\frac{1}{2}}). \tag{4.4}$$

Let $E^i = U^i - V^i$, with $E^0 = 0$. Then using (2.3a) and (4.4), we obtain

$$\partial_t E^n = a\Delta_h E^{n+\frac{1}{2}} + b\big(\varphi(U^{n+\frac{1}{2}}) - \varphi(V^{n+\frac{1}{2}})\big). \tag{4.5}$$

The proof of uniqueness is by using the induction method. Now, supposing $E^n = 0$ and taking an inner product of (4.5) with $E^{n+\frac{1}{2}}$, we obtain

$$\frac{1}{2k}\big(\|E^{n+1}\|_h^2 - \|E^n\|_h^2\big) = -a|E^{n+\frac{1}{2}}|_{1,h}^2 + b\big(\varphi(U^{n+\frac{1}{2}}) - \varphi(V^{n+\frac{1}{2}}), E^{n+\frac{1}{2}}\big)_h.$$

Taking the real part, we have

$$\frac{1}{2k}\big(\|E^{n+1}\|^2 - \|E^n\|_h^2\big) = -\alpha|E^{n+\frac{1}{2}}|_{1,h}^2 + \mathrm{Re}\big[b\big(\varphi(U^{n+\frac{1}{2}}) - \varphi(V^{n+\frac{1}{2}}), E^{n+\frac{1}{2}}\big)_h\big].$$

It follows from (1.3) that

$$\frac{1}{2k}\big(\|E^{n+1}\|_h^2 - \|E^n\|_h^2\big) \le |b|.\big|\big(\varphi(U^{n+\frac{1}{2}}) - \varphi(V^{n+\frac{1}{2}}), E^{n+\frac{1}{2}}\big)_h\big|. \tag{4.6}$$

Using the inequality

$$\||z_1|^2 z_1 - |z_2|^2 z_2\| \le (|z_1| + |z_2|)^2 |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}, \tag{4.7}$$

and applying Theorem 3, and (4.7), we get

$$\big|\big(\varphi(U^{n+\frac{1}{2}}) - \varphi(V^{n+\frac{1}{2}}), E^{n+\frac{1}{2}}\big)_h\big| \le 4\widetilde{C}^2 \|E^{n+\frac{1}{2}}\|_h^2. \tag{4.8}$$

Using (4.6) and (4.8), we obtain

$$\|E^{n+1}\|_h^2 - \|E^n\|_h^2 \le 8k\widetilde{C}^2|b|\|E^{n+\frac{1}{2}}\|_h^2,$$

from which, for k sufficiently small, we have

$$\|E^{n+1}\|_h \le \sqrt{\frac{1+4k\widetilde{C}^2|b|}{1-4k\widetilde{C}^2|b|}}\|E^n\|_h.$$

This implies that $E^{n+1} = 0$. The uniqueness follows. □

5 Convergence

In this section, we shall show that the approximate solution U^n of (2.3) converges to the exact solution u^n of (1.2). For this reason, we need the following Lemma.

Lemma 3 *Assume that the solution $u(x,t)$ of (1.2) is sufficiently regular and $u_0 \in H_0^1(0,1)$. Then, there exists a positive constant $\hat{C}$ independent of h and k such that*

$$|\varphi(u^{n+\frac{1}{2}}) - \varphi(U^{n+\frac{1}{2}})| \le \hat{C}|u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}|.$$

Proof It follows from Theorem 3 and (4.7) that

$$\begin{aligned}|\varphi(u^{n+\frac{1}{2}}) - \varphi(U^{n+\frac{1}{2}})| &\le \big(|u^{n+\frac{1}{2}}| + |U^{n+\frac{1}{2}}|\big)^2 |u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}| \\ &\le \hat{C}|u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}|, \end{aligned} \tag{5.1}$$

where $\hat{C} = (\check{C} + \widetilde{C})^2$ and $\check{C} = \max_{0\le x\le 1,\ 0\le t\le T} |u(x,t)|$. □

Theorem 5 *Suppose that the solution $u(x,t)$ of (1.2) is sufficiently smooth and the conditions (1.3)–(1.5) are fulfilled. Then, for k small enough, the solution of the difference scheme (2.3) converges to the solution of (1.2) in the discrete L^∞-norm and the rate of convergence is $O(h^2 + k^2)$.*

Proof Using Taylor expansion, we obtain

$$\partial_t u_j^n = a\Delta_h u_j^{n+\frac{1}{2}} + b\varphi\big(u_j^{n+\frac{1}{2}}\big) + r_j^n, \quad 0 \le n \le N-1,\ 1 \le j \le J, \tag{5.2a}$$

$$u_j^0 = u_0(x_j), \quad 1 \le j \le J, \tag{5.2b}$$

where $r_j^n \in S$ are the consistency errors of the difference scheme (2.3) and there exists a positive constant C such that (see [1, 12, 15, 18, 19])

$$\max_{j,n} |r_j^n| \le C(h^2 + k^2), \quad 0 \le n \le N-1,\ 1 \le j \le J. \tag{5.3}$$

Let $e_j^n = u_j^n - U_j^n$, $0 \le n \le N$, $1 \le j \le J$.

Substracting (2.3) from (5.2), we find

$$\begin{aligned}\partial_t e_j^n = a\Delta_h e_j^{n+\frac{1}{2}} + b\big(\varphi(u_j^{n+\frac{1}{2}}) - \varphi(U_j^{n+\frac{1}{2}})\big) + r_j^n, \quad 0 \le n \le N-1,\\ 1 \le j \le J, \end{aligned} \tag{5.4a}$$

$$e_j^0 = 0, \quad 1 \le j \le J. \tag{5.4b}$$

Forming the discrete inner product of (5.4a) with $e^{n+\frac{1}{2}}$ and taking the real part, Lemma 3 and (1.3) yield

$$\frac{1}{2k}\big(\|e^{n+1}\|_h^2 - \|e^n\|_h^2\big) \le |b|^2\hat{C}\|e^{n+\frac{1}{2}}\|_h^2 + \frac{1}{2}\|r^n\|_h^2 + \frac{1}{2}\|e^{n+\frac{1}{2}}\|_h^2.$$

Using (5.3), we have

$$(1-Ck)\|e^{n+1}\|_h^2 \le (1+Ck)\|e^n\|_h^2 + Ck(h^2+k^2)^2.$$

Applying the discrete Gronwall's inequality for small k so that $1-Ck>0$, we obtain for some positive constant C

$$\|e^n\|_h \le C(h^2+k^2), \quad 1\le n\le N. \tag{5.5}$$

Now multiplying (5.4a) by $\frac{1}{a}$ and forming its inner product with $\partial_t e^n$ and taking real parts, we obtain

$$\frac{\alpha}{|a|^2}\|\partial_t e^n\|_h^2 = -\frac{1}{2}\partial_t|e^n|_{1,h}^2 + \mathrm{Re}\left[\frac{b}{a}\left(\varphi(u_j^{n+\frac{1}{2}})-\varphi(U_j^{n+\frac{1}{2}}), \partial_t e^n\right)_h\right] + \mathrm{Re}\left[\frac{1}{a}(r^n, \partial_t e^n)_h\right].$$

It follows from Lemma 3 that

$$\begin{aligned}\frac{\alpha}{|a|^2}\|\partial_t e^n\|_h^2 &\le -\frac{1}{2}\partial_t|e^n|_{1,h}^2 + \frac{|b|}{|a|}\hat{C}\|e^{n+\frac{1}{2}}\|_h\|\partial_t e^n\|_h + \frac{1}{|a|}\|r^n\|_h\|\partial_t e^n\|_h\\ &\le -\frac{1}{2}\partial_t|e^n|_{1,h}^2 + \frac{\alpha}{2|a|^2}\|\partial_t e^n\|_h^2 + \frac{|b|^2\hat{C}^2}{2\alpha}\|e^{n+\frac{1}{2}}\|_h^2\\ &\quad + \frac{1}{2\alpha}\|r^n\|_h^2 + \frac{\alpha}{2|a|^2}\|\partial_t e^n\|_h^2.\end{aligned}$$

Therefore, from (5.3), we find

$$\frac{1}{2k}\left(|e^{n+1}|_{1,h}^2 - |e^n|_{1,h}^2\right) \le C\left[\|e^{n+\frac{1}{2}}\|_h^2 + (h^2+k^2)^2\right].$$

Using (5.5), we obtain

$$\frac{1}{2k}\left(|e^{n+1}|_{1,h}^2 - |e^n|_{1,h}^2\right) \le C(h^2+k^2)^2.$$

This implies that

$$|e^n|_{1,h} \le C(T)(h^2+k^2). \tag{5.6}$$

Applying Lemma 2, (5.5), and (5.6), we have

$$\|e^n\|_{\infty,h} \le C(h^2+k^2). \tag{5.7}$$

This completes the proof. □

6 Numerical implementation

Example 6.1 (The Kuramoto-Tsuzuki equation) In order to check theoretical properties in the previous section numerically, we consider the inhomogeneous equation

$$\frac{\partial u}{\partial t} = (1+i)\frac{\partial^2 u}{\partial x^2} - (1+i)|u|^2 u + \psi(x,t), \quad 0 < x < 1,\ 0 < t \le T, \tag{6.1a}$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \le T, \tag{6.1b}$$

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1, \tag{6.1c}$$

where

$$\psi(x,t) = (1+i)\exp\left(-\frac{(i+1)t}{10}\right).\sin(\pi x)\left[-\frac{1}{10} + \pi^2 \exp\left(-\frac{2t}{10}\right)\sin^2(\pi x)\right],$$

and whose exact solution is $u(x,t) = \exp(-\frac{(i+1)t}{10}).\sin(\pi x)$.

Discretize equations (6.1a)–(6.1c), using the finite difference method as (2.3a)–(2.3b), will allow us to solve the system of difference scheme by the Newton's method. Table 1 shows the estimate of error $\|e^n\|_{\infty,h} = \max_{j,n} |U_j^n - u(x_j,t_n)|$ for various step sizes h and k. According to Table 1, we can see that the computational results are getting better as h and k become smaller. Therefore, Table 1 confirms the second order convergence in L^∞-norm of the difference scheme for the Kuramoto-Tsuzuki equation. In Fig. 1 the surface shows the numerical solution $|U|$ of the Kuramoto-Tsuzuki equation (6.1) from $t = 0$ to $t = 10$.

Example 6.2 (Reaction diffusion equation) Computed by the difference scheme (2.3a)–(2.3b) the following inhomogeneous reaction diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - |u|^2 u + \varphi(x,t), \quad 0 < x < 1,\ 0 < t \le T, \tag{6.2a}$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \le T, \tag{6.2b}$$

Table 1 Error estimates and convergence ratios of the Kuramoto-Tsuzuki equation (6.1): $\Omega = (0,1)$, $T = 10$, $u_0(x) = \sin \pi x$

$J+1$	N	$\|e\|_{\infty,h} = \max_{j,n} \|U_j^n - u(x_j,t_n)\|$	$\|e\|_{\infty,h}/(h^2+k^2)$
10	100	0.0077	0.3828
20	200	0.0018	0.3607
40	400	4.4473×10^{-4}	0.3558
80	800	1.1082×10^{-4}	0.3546
160	1600	2.7682×10^{-5}	0.3543
320	3200	6.9192×10^{-6}	0.3543

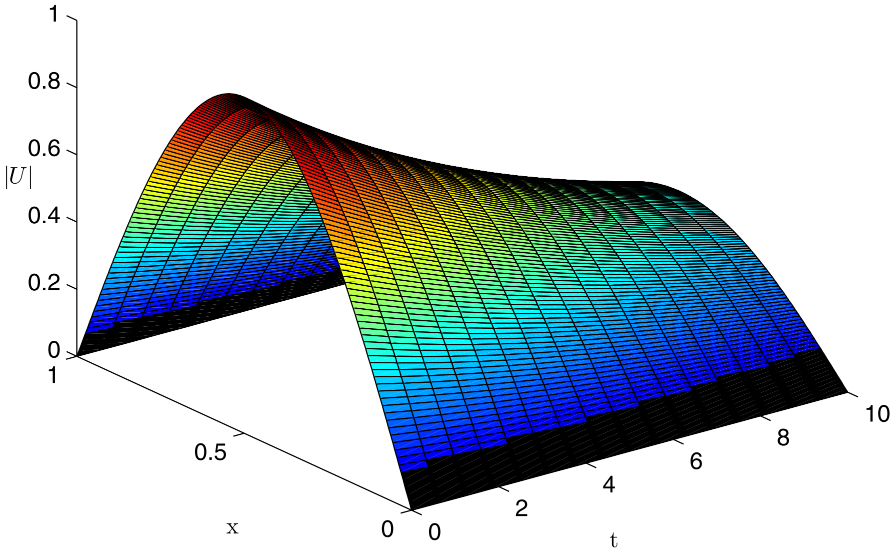

Fig. 1 The surface shows the numerical solution $|U|$ of the Kuramoto-Tsuzuki equation (6.1), when $k = h = \frac{1}{160}$ and $t \in [0, 10]$

Table 2 Error estimates and convergence ratios of the reaction diffusion equation (6.2): $\Omega = (0, 1)$, $T = 10$, $u_0(x) = \exp(i + 1). \sin \pi x$

$J + 1$	N	$\|e\|_{\infty,h} = \max_{j,n} \|U_j^n - u(x_j, t_n)\|$	$\|e\|_{\infty,h}/(h^2 + k^2)$
10	100	0.0074	0.3700
20	200	0.0019	0.3702
40	400	4.6278×10^{-4}	0.3702
80	800	1.1570×10^{-4}	0.3702
160	1600	2.8925×10^{-5}	0.3702
320	3200	7.2312×10^{-6}	0.3702

$$u(x, 0) = \exp(i + 1). \sin(\pi x), \quad 0 \le x \le 1, \tag{6.2c}$$

where

$$\varphi(x, t) = \exp\left((i + 1)\left(\frac{t}{5} - 1\right)^2\right)\left[\frac{2}{5}\left(\frac{t}{5} - 1\right) + \pi^2 + \exp\left(2\left(\frac{t}{5} - 1\right)^2\right). \sin^2(\pi x) + \frac{2i}{5}\left(\frac{t}{5} - 1\right)\right]. \sin(\pi x)$$

and whose exact solution is $u(x, t) = \exp((i + 1)(\frac{t}{5} - 1)^2). \sin(\pi x)$.

At each time level, the system of the nonlinear algebraic equations is solved by Newton's iterative method. Table 2 gives the maximum absolute errors of the difference solution at all grid points for different h and k. Moreover, the difference scheme

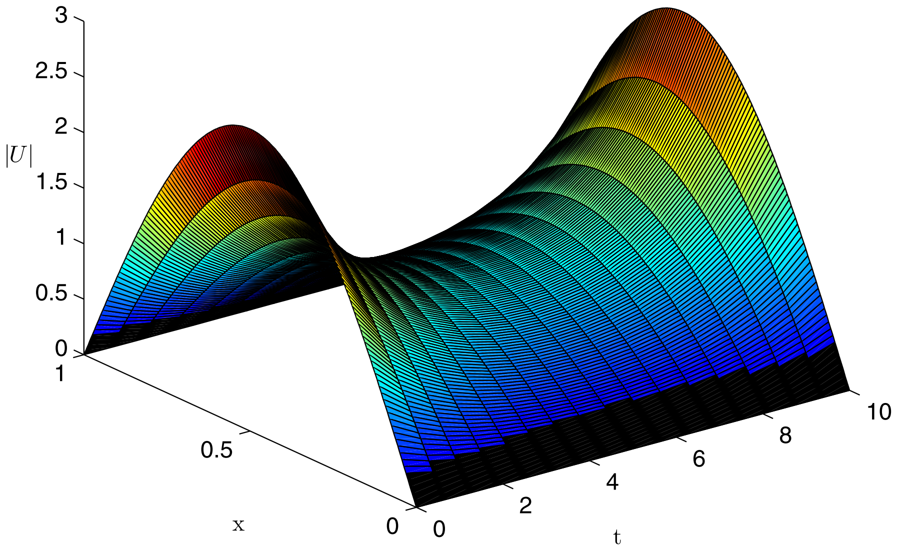

Fig. 2 The surface shows the numerical solution $|U|$ of the reaction diffusion equation (6.2), when $k = h = \frac{1}{160}$ and $t \in [0, 10]$

has a second order convergence in L^∞-discrete norm as shown in Table 2. In Fig. 2 the surface shows the numerical solution $|U|$ of the reaction diffusion equation (6.2) from $t = 0$ to $t = 10$.

Remark The results in the last columns of Table 1 and Table 2 show that the theoretical error estimate, $\|e^n\|_{\infty,h} \leq C(h^2 + k^2)$ is optimal as they give, for each numerical example, the smallest constant C.

Acknowledgement The authors are grateful to the anonymous referees for several suggestions and comments on this article.

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